Conformal Field Theory and Gravity

Solutions to Problem Set 8 — Fall 2024

1. Radial Quantization of the Free Scalar

(a) Let's introduce some notation. Let

$$ds^2_{S^{d-1}} = h_{ij}(\theta^k)d\theta^i d\theta^j$$

be the metric of the unit sphere S^{d-1}. For instance, in $d = 2, 3$ we have

$$ds^2_{S^1} = d\phi^2, \quad ds^2_{S^2} = d\theta^2 + \sin^2(\theta)d\phi^2.$$

The corresponding volume element is $\sqrt{h}\prod_i d\theta^i \equiv d\Omega$, to connect with the notation from the exercise. The metric in radial coordinates (r, θ^i) resp. cylinder coordinates (σ, θ^i) coordinates reads

$$ds^2 = dr^2 + r^2 ds^2_{S^{d-1}} = e^{2\sigma}\left[d\sigma^2 + ds^2_{S^{d-1}}\right]$$

since $dr = d(e^\sigma) = e^\sigma d\sigma$. In particular, in the (σ, θ^i) coordinates we have $\sqrt{g} = \exp(d\sigma)\sqrt{h}$ so the volume element is $e^{d\sigma}d\sigma d\Omega$.

After throwing away a boundary term, the action is given by

$$S[\phi] = \int d^d x \frac{1}{2}(-\phi\Box\phi), \quad \Box = \partial_\mu\partial^\mu$$

working in Cartesian coordinates x^μ. In a general coordinate system, the Laplacian acting on a general function f is instead

$$\Box f = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu f)$$

Evaluating the Laplacian in cylinder coordinates (σ, θ^i) and plugging it into the above relation will yield the correct answer. Nevertheless, we will use a different strategy. The Euclidean action can be rewritten in a purely covariant form as

$$S_E = \int d^d x \frac{1}{2}(\partial_\mu\phi)^2 = \frac{1}{2}\int d^d x \sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

We can now just use our change of variables as

$$S_E = \frac{1}{2}\int d\sigma \int d^{d-1}\theta\sqrt{h}e^{(d-2)\sigma}\left[(\partial_\sigma\phi)^2 + h^{ij}\partial_i\phi\partial_j\phi\right]$$

$$= \frac{1}{2}\int d\sigma d\Omega e^{(d-2)\sigma}\left[(\partial_\sigma\phi)^2 + h^{ij}\partial_i\phi\partial_j\phi\right]$$

According to the exercise, we should define a field $\phi = \exp\left(-\frac{1}{2}(d-2)\sigma\right)\chi$. With this definition, we have

$$\partial_\sigma \phi = \frac{2-d}{2} e^{-\frac{d-2}{2}\sigma}\chi + e^{-\frac{d-2}{2}\sigma}\partial_\sigma \chi$$

Plugging that into the action and neglecting total derivatives yields the answer from the exercise.

Using the other strategy, we obtain

$$-e^{d\sigma}\phi\Box\phi = \chi D\chi, \quad D = -\frac{\partial^2}{\partial\sigma^2} + \left(\frac{d-2}{2}\right)^2 - \Box_{S^{d-1}}.$$

Therefore

$$S[\phi] = \frac{1}{2}\int d\sigma d\Omega \left[(\partial_\sigma\chi)^2 - \chi\Box_{S^{d-1}}\chi + \left(\frac{d-2}{2}\right)^2\chi^2\right].$$

If we write $-\chi\Box_{S^{d-1}}\chi = \partial_i\chi\partial^i\chi$, we recover the result from the exercise.

(b) **Solutions of the Equation $D\chi = 0$**

Suppose that $Y(\theta^i)$ is an eigenfunction of the Laplacian $\Box_{S^{d-1}}$ in d dimensions, with eigenvalue $-\gamma$. If we take the Ansatz

$$\chi(\sigma, \theta^i) = e^{\pm\omega\sigma}Y(\theta^i),$$

then

$$D\chi = \left[-\omega^2 + \left(\frac{d-2}{2}\right)^2 + \gamma\right]\chi \implies \omega = \sqrt{\gamma + \left(\frac{d-2}{2}\right)^2}.$$

If you have studied the theory of spherical harmonics, you may know that in d dimensions, the spherical harmonics of S^{d-1} have eigenvalues $\gamma = \ell(\ell+d-2)$, so the possible values of ω are

$$\omega_\ell = \sqrt{\ell(\ell+d-2) + \left(\frac{d-2}{2}\right)^2} = \ell + \frac{d-2}{2}.$$

In particular, for $d = 3$ the allowed energies are $\pm\omega_\ell = \pm(\ell + \frac{1}{2})$.

From now on, let us fix $d = 3$. The most general solution is a sum of all solutions of the form (22), namely

$$\chi(\sigma, \theta, \phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(b^+_{\ell,m}e^{\omega_\ell\sigma} + b^-_{\ell,m}e^{-\omega_\ell\sigma}\right)Y_{\ell,m}(\theta,\phi),$$

where $b^\pm_{\ell,m}$ are some operators acting on Hilbert space. The reality condition — using $Y^*_{\ell,m} = (-1)^m Y_{\ell,-m}$ — is

$$\chi(\sigma,\theta^i) = \chi(-\sigma,\theta^i)^\dagger = \sum_{\ell,m}(-1)^m\left[(b^+_{\ell,m})^\dagger e^{-\omega_\ell\sigma} + (b^-_{\ell,m})^\dagger e^{\omega_\ell\sigma}\right]Y_{\ell,-m}(\theta,\phi)$$

$$= \sum_{\ell,m} \left[(-1)^m (b^-_{\ell,-m})^\dagger e^{\omega_\ell \sigma} + (-1)^m (b^+_{\ell,-m})^\dagger e^{-\omega_\ell \sigma}\right] Y_{\ell,m}(\theta,\phi)$$

(relabeling $m \to -m$ in passing from the first to the second line), so by imposing that this agrees with (25) for all σ and θ, ϕ, we learn that

$$b^+_{\ell,m} = (-1)^m (b^-_{\ell,-m})^\dagger.$$

It is more natural to redefine the modes as

$$a^+_{\ell,m} = \sqrt{2\omega_\ell}\, b^+_{\ell,m}, \quad a^-_{\ell,m} = \sqrt{2\omega_\ell}(-1)^m b^-_{\ell,-m}$$

such that

$$a^+_{\ell,m} = (a^-_{\ell,m})^\dagger.$$

This also leads to a more standard expression for the field χ, namely

$$\chi(\sigma,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{\sqrt{2\omega_\ell}} \left[a^+_{\ell,m} e^{\omega_\ell \sigma} Y_{\ell,m}(\theta,\phi) + a^-_{\ell,m} e^{-\omega_\ell \sigma} Y_{\ell,m}(\theta,\phi)^*\right]$$

which is now manifestly hermitian.

(c) **Computation of the Hamiltonian H**

We are left to compute H, by integrating

$$H = \frac{1}{2} \int_{S^2} d\Omega \left[-(\partial_\sigma \chi)^2 + \chi \left(-\Box_{S^2} + \frac{1}{4}(d-2)^2\right) \chi\right].$$

For the first term, notice that

$$\partial_\sigma \chi = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sqrt{\frac{\omega_\ell}{2}} \left[a^+_{\ell,m} e^{\omega_\ell \sigma} Y_{\ell,m}(\theta,\phi) - a^-_{\ell,m} e^{-\omega_\ell \sigma} Y_{\ell,m}(\theta,\phi)^*\right]$$

such that

$$-\int_{S^2} (\partial_\sigma \chi)^2 = \sum_{\ell,m} \frac{\omega_\ell}{2} \left[-(-1)^m a^+_{\ell,m} a^+_{\ell,-m} e^{2\omega_\ell \sigma} - (-1)^m a^-_{\ell,m} a^-_{\ell,-m} e^{-2\omega_\ell \sigma} + a^+_{\ell,m} a^-_{\ell,m} + a^-_{\ell,m} a^+_{\ell,m}\right].$$

To obtain this, we have used the orthogonality condition from the exercise. For the second part, we have instead

$$\left(-\Box_{S^2} + \frac{1}{4}(d-2)^2\right) \chi = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\omega_\ell^{3/2}}{\sqrt{2}} \left[a^+_{\ell,m} e^{\omega_\ell \sigma} Y_{\ell,m}(\theta,\phi) + a^-_{\ell,m} e^{-\omega_\ell \sigma} Y_{\ell,m}(\theta,\phi)^*\right]$$

so

$$\int_{S^2} \chi \left(-\Box_{S^2} + \frac{1}{4}(d-2)^2\right) \chi = \sum_{\ell,m} \frac{\omega_\ell}{2} \left[(-1)^m a^+_{\ell,m} a^+_{\ell,-m} e^{2\omega_\ell \sigma} + (-1)^m a^-_{\ell,m} a^-_{\ell,-m} e^{-2\omega_\ell \sigma} + a^+_{\ell,m} a^-_{\ell,m} +\right.$$

Adding both contributions, we see that the $a^+ a^+$ and $a^- a^-$ terms cancel, but the $a^+ a^-$ and $a^- a^+$ terms survive, and

$$H = \sum_{\ell,m} \frac{\omega_\ell}{2} \left[a^+_{\ell,m} a^-_{\ell,m} + a^-_{\ell,m} a^+_{\ell,m}\right].$$

In particular, we point to the fact that H does not depend on the time coordinate σ.

(d) **Computation of the Commutator** $[\chi, \partial_\sigma \chi]$

Since we have explicit expressions for χ and $\partial_\sigma \chi$, let us compute the commutator explicitly:

$$[\chi, \partial_\sigma \chi] = \frac{1}{2} \sum_{\ell,m} \sum_{\ell',m'} \sqrt{\frac{\omega_{\ell'}}{\omega_\ell}} \left[[a^+_{\ell,m}, a^+_{\ell',m'}] e^{(\omega_\ell + \omega_{\ell'})\sigma} Y_{\ell,m} Y_{\ell',m'} \right.$$

$$- [a^+_{\ell,m}, a^-_{\ell',m'}] e^{(\omega_\ell - \omega_{\ell'})\sigma} Y_{\ell,m} Y^*_{\ell',m'} + [a^-_{\ell,m}, a^+_{\ell',m'}] e^{(-\omega_\ell + \omega_{\ell'})\sigma} Y^*_{\ell,m} Y_{\ell',m'}$$

$$\left. - [a^-_{\ell,m}, a^-_{\ell',m'}] e^{-(\omega_\ell + \omega_{\ell'})\sigma} Y^*_{\ell,m} Y^*_{\ell',m'} \right].$$

Since this must equal (14) from the exercise, we see in particular that the RHS cannot depend on σ. So we must have $[a^+_{\ell,m}, a^+_{\ell',m'}] = 0$, and by taking the hermitian conjugate of that also $[a^-_{\ell,m}, a^-_{\ell',m'}] = 0$. Looking at the second and third terms, we see that in addition we must have

$$[a^+_{\ell,m}, a^-_{\ell',m'}] \propto \delta_{\ell,\ell'}.$$

Using this, we rewrite the above equation as

$$[\chi, \partial_\sigma \chi] = \sum_{\ell,m,m'} [a^-_{\ell,m'}, a^+_{\ell,m}] Y_{\ell,m}(\theta, \phi) Y_{\ell,m'}(\theta, \phi)^*.$$

After relabeling the dummy indices m, m' in one of the terms and using the property that $[A, B] = -[B, A]$, we get

$$[\chi, \partial_\sigma \chi] = \sum_{\ell,m} c(\ell, m) Y_{\ell,m}(\theta, \phi) Y_{\ell,m}(\theta, \phi)^*,$$

where $c(\ell, m)$ are some constants. Using the orthogonality relation for spherical harmonics,

$$\sum_{m=-\ell}^{\ell} Y_{\ell,m}(\theta, \phi) Y_{\ell,m}(\theta, \phi)^* = 1,$$

we find that

$$[\chi, \partial_\sigma \chi] = \sum_{\ell} c(\ell, m).$$

Matching this with the result from the exercise (equation 14), we conclude that $c(\ell, m) = 1$ for all ℓ, m, so

$$[a^-_{\ell,m}, a^+_{\ell',m'}] = \delta_{\ell,\ell'} \delta_{m,m'}.$$

(e) **Hamiltonian H in Normal-Ordered Form**

Returning to our expression for the Hamiltonian H, we get

$$H = \sum_{\ell,m} \frac{\omega_\ell}{2} \left(a^+_{\ell,m} a^-_{\ell,m} + [a^-_{\ell,m}, a^+_{\ell,m}] + a^-_{\ell,m} a^+_{\ell,m} \right).$$

Since $[a^-_{\ell,m}, a^+_{\ell,m}] = 1$, we have

$$H = \sum_{\ell,m} \omega_\ell a^+_{\ell,m} a^-_{\ell,m} + E_0,$$

where

$$E_0 = \sum_{\ell,m} \frac{\omega_\ell}{2} = \sum_{\ell=0}^{\infty} (\ell + \frac{1}{2})^2 = \infty.$$

The term E_0 is a divergent constant that we subtract from H. This subtraction is equivalent to normal-ordering H.

The Hamiltonian H becomes an infinite sum of harmonic oscillators, each associated with a creation operator $a^+_{\ell,m}$ and an annihilation operator $a^-_{\ell,m}$. To define our Hilbert space, we start with a vacuum state $|\Omega\rangle$ that obeys

$$\forall \ell, m : a^-_{\ell,m} |\Omega\rangle = 0.$$

We build excited states by acting with the creation operators $a^+_{\ell,m}$. For instance, the state $|\ell, m\rangle = a^+_{\ell,m}|\Omega\rangle$ has energy

$$H|\ell, m\rangle = [H, a^+_{\ell,m}]|\Omega\rangle + a^+_{\ell,m} H|\Omega\rangle = \omega_\ell |\ell, m\rangle.$$

A complete basis of states in the theory is given by

$$|\Psi\rangle = \prod_{\ell=0}^{\infty} \prod_{m=-\ell}^{\ell} (a^+_{\ell,m})^{n_{\ell,m}} |\Omega\rangle,$$

where $n_{\ell,m} \in \mathbb{N}$. The state $|\Psi\rangle$ has energy

$$H|\Psi\rangle = E(\Psi)|\Psi\rangle, \quad E(\Psi) = n_{0,0}\omega_0 + \sum_{m=-1}^{1} n_{1,m}\omega_1 + \ldots + \sum_{m=-\ell}^{\ell} n_{\ell,m}\omega_\ell + \ldots$$

or equivalently

$$E(\Psi) = \sum_{\ell=0}^{\infty} n_\ell (\ell + \frac{1}{2}), \quad n_\ell = \sum_{m=-\ell}^{\ell} n_{\ell,m}.$$

(f) **Matching with Flat-Space Theory**

In the flat-space theory, the free scalar $\phi(x)$ has dimension $\Delta_\phi = \frac{d-2}{2} = \frac{1}{2}$, setting $d \to 3$. Its descendants

$$\partial_{\mu_1} \cdots \partial_{\mu_n} \phi(x)$$

have dimension $\Delta = \Delta_\phi + n = n + \frac{1}{2}$. These scaling dimensions match the one-particle states

$$|\ell, m\rangle = a^+_{\ell,m}|\Omega\rangle, \quad E = \ell + \frac{1}{2},$$

if we set $\ell = n$. The number of states with energy $n + \frac{1}{2}$ also agrees. For $n = 1$, there are three descendants $\partial_\mu \phi$, and at the same time, there are three states with energy $\frac{3}{2}$, namely $|1, -1\rangle, |1, 0\rangle, |1, 1\rangle$. At the next level $n = 2$, there are 5 different descendants: there are $\frac{1}{2}d(d + 1) = 6$ operators of the form $\partial_\mu \partial_\nu \phi$, but

$$\Box \phi = \sum_{\mu=1}^{3} \partial_\mu^2 \phi = 0,$$

so the number of independent operators is only $6-1=5$. Similarly, there are $2\ell+1=5$ states of the form $|2,m\rangle$, with $m=-2,\ldots,2$. This reasoning continues for higher levels. We can also consider operators with multiple fields, such as $O =: \phi^2 :$ with dimension $\Delta_O = 2\Delta_\phi = 1$. This operator corresponds to a two-particle state $(a^+_{0,0})^2|\Omega\rangle$ with energy $2\omega_0 = 1$. Using similar logic, we can match a general operator of the schematic form

$$O =: \partial^{j_1}\phi\,\partial^{j_2}\phi\,\cdots\,\partial^{j_n}\phi :$$

with scaling dimension $\Delta_O = \frac{n}{2}+j_1+j_2+\cdots+j_n$ to an n-particle state with energy $E=\Delta_O$.

(g) **Computation of the Integral** $F(r)$

We are instructed to compute the integral

$$F(r) = \int_{S^{d-1}} d\Omega\, r^{d-1} x^\mu r j_\mu(x) = -\int_{S^{d-1}} d\Omega\, r^{d-2} x^\mu x^\nu T_{\mu\nu}(x)$$

with

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial_\rho\phi)^2 + \xi(g_{\mu\nu}\Box - \partial_\mu\partial_\nu)\phi^2.$$

Let us work in flat space, with coordinates $\{\tau, n^\mu\}$ defined by $x^\mu = e^\tau n^\mu$. In particular,

$$x^\mu\partial_\mu f(x) = \partial_\tau f(\tau,n), \quad x^\mu x^\nu\partial_\mu\partial_\nu = \partial_\tau^2 - \partial_\tau.$$

Therefore,

$$x^\mu x^\nu T_{\mu\nu} = \frac{1}{2}(\partial_\tau\phi)^2 - \frac{1}{2}h^{ij}\partial_i\phi\partial_j\phi + \xi\left[e^{2\tau}\Box - \partial_\tau^2 + \partial_\tau\right]\phi^2,$$

where we use that

$$ds^2 = e^{2\tau}\left[d\tau^2 + h_{ij}d\theta^i d\theta^j\right].$$

Now let us perform the field redefinition $\phi(\tau,n) = e^{-\frac{1}{2}(d-2)\tau}\chi(\tau,n)$ as before. Moreover, from (53) it follows that

$$e^{2\tau}\Box = \Box_{S^{d-1}} + \partial_\tau^2 + (d-2)\partial_\tau$$

when acting on scalar functions. The first term in $x^\mu x^\nu T_{\mu\nu}$ transforms as

$$e^{(d-2)\tau}\frac{1}{2}(\partial_\tau\phi)^2 = \frac{1}{2}(\partial_\tau\chi)^2 + \frac{1}{2}\left(\frac{d-2}{2}\right)^2\chi^2 - \frac{d-2}{4}\partial_\tau(\chi^2),$$

whereas the last term gives

$$e^{(d-2)\tau}\left(e^{2\tau}\Box - \partial_\tau^2 + \partial_\tau\right)\phi^2 = [-(d-1)(d-2) + (d-1)\partial_\tau + \Box_{S^{d-1}}]\chi^2.$$

If $\xi = \xi_c$, the full stress tensor simplifies to

$$\xi = \xi_c: \quad -e^{(d-2)\tau}x^\mu x^\nu T_{\mu\nu} = -\frac{1}{2}(\partial_\tau\chi)^2 + \frac{1}{2}\chi(-\Box_{S^{d-1}}\chi) + \frac{1}{2}\left(\frac{d-2}{2}\right)^2\chi^2,$$

neglecting a boundary term $\nabla_i(\ldots)$ that vanishes once integrated over the sphere. This expression matches the Hamiltonian density that we started with.

If $\xi \neq \xi_c$, there is an additional term:

$$-e^{(d-2)\tau}x^\mu x^\nu T_{\mu\nu} \supset (\xi - \xi_c)[(d-1)(d-2) - (d-1)\partial_\tau - \Box_{S^{d-1}}]\chi^2.$$

This contributes to H, although the term vanishes after integration over S^{d-1}. The remaining terms contribute as a mass term and an interaction term $\chi\partial_\tau\chi \sim \chi\Pi$.

2. Unitarity bounds

(a) Let us illustrate the solution with the scalar. As you've seen in the lecture.

$$\langle O|K_\mu P_\nu|O\rangle = 2\Delta\delta_{\mu\nu}\langle O|O\rangle = 2\Delta\delta_{\mu\nu}. \tag{1}$$

Thus, the norm of the state $|z\rangle = z_\mu P^\mu|O\rangle$ is $\langle z|z\rangle = 2\Delta|z|^2$. Since this must be positive, we conclude that $\Delta \geq 0$, with $\Delta = 0$ only possible when $P_\mu|O\rangle = 0$. This represents a state without descendants, meaning that $[P_\mu, O(x)] = i\partial_\mu O(x) = 0$, so $O(x)$ does not depend on its position x^μ. Thus, $O(x)$ must be a constant operator. In fact, this implies that $O(x)$ is the identity operator, corresponding to the vacuum state (which transforms as a scalar primary with $\Delta = 0$).

(b) For tensor fields (point (a) and (c) of the exercise set), let us define

$$|A_\mu[k]\rangle := |A\rangle_{\mu\sigma_1\ldots\sigma_{\ell-1}}k^{\sigma_1\ldots\sigma_{\ell-1}}, \quad |k\rangle := P_\mu|A^\mu[k]\rangle, \tag{2}$$

where $k^{\sigma_1\ldots\sigma_{\ell-1}}$ can be assumed to be completely symmetric and traceless. We seek the consequences of $\langle k|k\rangle \geq 0$. Hence, we compute using the conformal algebra

$$0 \leq \langle k|k\rangle = \langle A_\mu[k]|K^\mu P^\nu|A_\nu[k]\rangle = 2\Delta\langle A_\mu[k]|A^\mu[k]\rangle - 2i\langle A_\mu[k]|M^{\mu\nu}|A_\nu[k]\rangle. \tag{3}$$

Next, we compute using the explicit definition of $M_{\mu\nu}$ that

$$M^{\mu\nu}|A_\nu[k]\rangle = -i(\ell + d - 2)|A_\mu[k]\rangle. \tag{4}$$

Hence,

$$0 \leq \langle k|k\rangle = 2\left[\Delta - \ell - d + 2\right]\langle A_\mu[k]|A^\mu[k]\rangle. \tag{5}$$

Since $\langle A_\mu[k]|A^\mu[k]\rangle \geq 0$, we conclude that $\Delta \geq \ell + d - 2$, which is the desired bound. If the norm of $|k\rangle$ vanishes, then $P_\mu|A_{\mu\sigma_1\ldots\sigma_{\ell-1}}\rangle = 0$. Repeating the previous argument, we conclude that

$$\partial_\mu A^{\mu\sigma_1\ldots\sigma_{\ell-1}}(x) = 0, \tag{6}$$

meaning that A is conserved. Indeed, we already saw that a Noether current has dimension $d-1$ and the stress tensor has dimension d; here we have proven the converse.

(c) Commuting $\hat{C}$ with Generators

The check that $\hat{C}$ commutes with all generators is done in Mathematica. To evaluate it in a primary state $|A_{\mu_1\ldots\mu_\ell}\rangle$ of dimension Δ and spin ℓ, we note that

$$K_\mu P^\mu = P_\mu K^\mu + 2dD, \tag{7}$$

since $\delta^{\mu\nu}M_{\mu\nu} = 0$ (as $M_{\mu\nu}$ is antisymmetric). Thus,

$$\hat{C}|\text{primary}\rangle = \left[D^2 - dD + C_{\mathrm{SO}(d)}\right]|\text{primary}\rangle. \tag{8}$$

In particular, if the primary has spin ℓ and dimension Δ, we find that

$$\hat{C}|A_{\mu_1\ldots\mu_\ell}\rangle = \left[\Delta(\Delta - d) + \ell(\ell + d - 2)\right]|A_{\mu_1\ldots\mu_\ell}\rangle. \tag{9}$$

Note: If desired, you can show that $\langle A_\mu[k]|A^\mu[k]\rangle = c_{\ell,d}k^*_{\mu_1\ldots\mu_{\ell-1}}k^{\mu_1\ldots\mu_{\ell-1}}$ for some positive coefficient $c_{\ell,d} > 0$ with a group-theoretical interpretation.